Atomic radiative corrections without QED: role of the zero-point field

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Abstract

We derive the atomic radiative corrections predicted by QED using an alternative approach that offers the advantage of physical clarity and transparency. The element that gives rise to these corrections is the fluctuating zero-point radiation field (ZPF) of average energy $\hbar\omega/2$ per mode, which —in contrast with QED— is taken here as a primordial real entity in permanent interaction with matter and responsible for its quantization. After briefly recalling how quantum mechanics itself emerges as a result of the balance between the zpf and radiation reaction, the most important higher-order effects of the radiative terms on the atom are studied. The nonrelativistic QED formulas for the lifetimes and the Lamb shift, as well as the corrections to the latter due to external factors that modify the vacuum field, are thus obtained in a self-consistent approach and without the need to resort to second quantization to the present order of approximation.

Keywords: Radiative corrections, Atomic Lamb shift, Atomic lifetimes, Zero-point field

1 Introduction

The random zero-point radiation field (ZPF) of mean energy $\hbar\omega/2$ per normal mode, taken as a real field, has been shown in a series of recent papers [1]-[3] to be responsible for the basic quantum properties of matter. In particular, the usual quantum description, as afforded, e.g. by the Schrödinger equation, is obtained as a result of reducing the original phase-space description of the entire particle-ZPF system to the configuration space of the particle in the time-asymptotic limit, in which an energy balance is reached between radi-

ation reaction and the ZPF and higher-order effects of the radiative terms can be neglected.

This paper is concerned with the main effects on the atom of the previously neglected radiative terms, calculated to lowest order in $\alpha = e^2/\hbar c$. After briefly reviewing some consequences of the energy-balance condition and the role of the ZPF in fixing the atomic stationary states, an analysis of the dynamics in the absence of energy balance is made, leading to formulas for the radiative lifetimes of excited states. Further, a calculation of the contribution of the radiative terms to the average energy gives the nonrelativistic formula for the Lamb shift. Finally, a modification of the background field through the presence of an external field or material objects is shown to produce in general a change in the radiative lifetimes and a shift of the atomic energy levels.

The nonrelativistic, spinless, electric dipole approximation is made throughout the paper. The correct results (i.e. those predicted by non-relativistic quantum electrodynamics, QED) are obtained in all cases. With these results we demonstrate that the theory of stochastic electrodynamics in its present form [1]-[3] takes us *beyond* quantum mechanics, to the realm of QED. Though most of the results for the radiative corrections derived in the present work are well known, their connection with the condition of energy balance between the ZPF and radiation reaction is not. An interesting point is that in each case explicit formulas for the effects discussed are obtained, along with a clear physical picture of their meaning

The theory used here should be clearly distinguished from what is called semiclassical theory (e.g. [4]). The SED approach is *not* an attempt to replace quantum physics with a classical (or semiclassical) theory; quite the contrary, it is an endeavour intended to give deep physical support to quantum theory by answering fundamental questions as, e.g., on the physical mechanism that leads to the quantization and the stability of the atom, or the physical cause and nature of the quantum fluctuations. In this approach the ZPF is seen to play a crucial role for atomic stability through the energy-balance condition; there is no quantization in the absence of this field. The already quantized atom continues to interact with the ZPF, which leads to the radiative corrections here studied. Although in the quantum regime also the field satisfies quantum rules, as discussed in [5], [6], these are not explicitly needed for the present purposes. Quantized matter under the action of the ZPF is sufficient to obtain the nonrelativistic radiative corrections to lowest order in α.

2 The quantum regime

2.1 Radiationless approximation

For clarity in the exposition, let us recall in this section the main steps leading to the Schrödinger equation on the basis of the existence of the ZPF. For details see [2], [3].

The motion of the particle is governed in the nonrelativistic limit by the

equations (we use one-dimensional notation wherever possible, for simplicity)

$$\dot{x} = p/m, \quad \dot{p} = f(x) + m\tau\dddot{x} + eE(t), \tag{1}$$

where $f(x)$ is the external force, $E(t)$ is the electric component of the random ZPF in the long-wavelength approximation and $m\tau\dddot{x}$ is the radiation reaction force in the Abraham-Lorentz approximation, with $\tau = 2e^2/3mc^3$ ($\approx 10^{-23}$ s for the electron). The density R of points in the particle's phase space is determined by

$$\frac{\partial}{\partial t}R + \frac{\partial}{\partial x}(\dot{x}R) + \frac{\partial}{\partial p}\left(f(x) + m\tau\dddot{x}\right)R = -\frac{\partial}{\partial p}E(t)R. \tag{2}$$

Averaging over the realizations of the field one obtains for the (mean) density in the phase space of the particle, $\overline{R(x,p,t)}^E \equiv Q(x,p,t)$, the generalized Fokker-Planck equation

$$\frac{\partial}{\partial t}Q + \hat{L}Q = e^2\frac{\partial}{\partial p}\hat{D}(t)Q, \tag{3}$$

with $\hat{L}$ the Liouville operator

$$\hat{L} = \frac{1}{m}\frac{\partial}{\partial x}p + \frac{\partial}{\partial p}(f + m\tau\dddot{x}), \tag{4}$$

and $\hat{D}$ the diffusion operator, given to first order in e^2 by

$$e^2\hat{D}(t)Q = e^2\int^t dt'\overline{E(t)E(t')}^E e^{-\hat{L}(t-t')}\frac{\partial Q}{\partial p}. \tag{5}$$

The correlation of the electric field components is related with the spectral energy density of the field through

$$\overline{E(t)E(t')}^E = (4\pi/3)\int_0^\infty \rho(\omega)\cos\omega(t-t')d\omega. \tag{6}$$

For the ZPF (the field at temperature $T = 0$) the spectral energy density is given by

$$\rho(\omega,T)_{T=0} = \rho_0(\omega) = \frac{\hbar\omega^3}{2\pi^2c^3}. \tag{7}$$

To make the transition from the phase-space equation (3) to a description in configuration space, the characteristic function $\widetilde{Q}(x,z,t) = \int Q(x,p,t)e^{ipz}dp$ is introduced, so that the marginal probability density is $\rho(x,t) = \int Q(x,p,t)dp = \widetilde{Q}(x,0,t)$. By expanding the Fourier transform of Eq. (3) into a power series around $z = 0$ and separating the coefficients of z^k ($k = 0,1,2,\ldots$), a hierarchy of coupled equations for moments of p of increasing order is obtained. The first two are the continuity equation and the equation for the transfer of momentum, which with the help of the change of variables

$$z_\pm = x \pm \eta z \tag{8}$$

are shown to lead, in the limit $z \to 0$ (when both z_+ and z_- reduce to x) and after some approximations, to the Schrödinger equation in terms of the parameter η (to be determined below)

$$-2\frac{\eta^2}{m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = 2i\eta\frac{\partial\psi}{\partial t} \tag{9}$$

and its complex conjugate, with

$$\rho(x,t) = \psi^*(x)\psi(x). \tag{10}$$

It is important to stress that Eq. (10) is an integral part of the theory; it is not a subsidiary postulate. This result indicates that a regime of unitary (time-reversible) evolution has been attained, in which the mechanical subsystem has acquired its quantum properties [2], [3].

2.2 The meaning of $\hbar$ in the Schrödinger equation

In order for (9) to be fully equivalent to the Schrödinger equation, the value of the parameter η appearing in it must be independent of the problem, and equal to $\hbar/2$. Although the calculation of η has been presented in previous work [3], [7], we briefly reproduce it here because it serves to disclose the precise point of entry of Planck's constant into the Schrödinger equation.[1]

The value of η will be determined by resorting to the energy-balance condition, which equates the average power lost by the particle through Larmor radiation, to the average power extracted by the particle from the random field. To establish the energy-balance condition we take the generalized Fokker-Planck equation (3), namely

$$\frac{\partial}{\partial t}Q + \frac{1}{m}\frac{\partial}{\partial x}pQ + \frac{\partial}{\partial p}(f + m\tau\dddot{x})Q = e^2\frac{\partial}{\partial p}\hat{D}(t)Q, \tag{11}$$

($p = m\dot{x}$) multiply it by p^2 and integrate over the entire particle phase space. Assuming all surface terms to vanish at infinity, we obtain

$$\frac{1}{2m}\frac{d}{dt}\left\langle p^2\right\rangle = \frac{1}{2m}\frac{d}{dt}\int p^2 Q dx dp = \frac{1}{m}\left\langle fp + m\tau p\dddot{x} - e^2 p\hat{D}\right\rangle,$$

where $\langle g\rangle = \int g(x,p)Qdxdp$. Since $d\left\langle V\right\rangle/dt = -\left\langle fp\right\rangle/m$, the total average energy gain or loss per unit time is given by

$$\frac{d}{dt}\left\langle H\right\rangle = \frac{d}{dt}\left\langle \frac{1}{2m}p^2 + V\right\rangle = m\tau\left\langle \dot{x}\,\ddot{x}\right\rangle - \frac{e^2}{m}\left\langle p\hat{D}\right\rangle, \tag{12}$$

where H is the mechanical Hamiltonian function.. The first term on the right-hand side represents the average power dissipated by the particle through Larmor radiation; the second term represents the average power extracted by the

[1] In addition, the present calculation serves to correct a factor 1/2 mistakenly introduced in previously published versions of Eq. (12) and the following.

particle from the ZPF and absorbed by the momentum fluctuations. For energy balance to hold in the mean, these terms must compensate each other, i.e.,

$$m\tau \left\langle \dot{x}\, \dddot{x} \right\rangle = \frac{e^2}{m} \left\langle p\hat{D} \right\rangle . \tag{13}$$

For a calculation of these two average values to lowest order in $\tau \sim e^2$ we use the solutions of Eq. (9) containing the parameter η. Since the ZPF represents the background field in its ground state, the particle must also be in its ground state (denoted with the subindex 0). For the left-hand side of Eq. (13) this leads to

$$m\tau \left\langle \dot{x}\, \dddot{x} \right\rangle_0 = -m\tau \sum_k \omega_{0k}^4 \left| x_{0k} \right|^2 , \tag{14}$$

where $\omega_{0k} = \left(\mathcal{E}_0 - \mathcal{E}_k \right) / 2\eta$, $x_{0k} = \int \psi_0^* x \psi_k dx$, $\mathcal{E}_k$ are the energy eigenvalues and ψ_k the corresponding eigenfunctions. For the calculation of the right-hand side of Eq. (13), which is somewhat more elaborate, we introduce Eq. (7) for the spectral energy density of the field —which is proportional to $\hbar$— into Eq. (5), whence

$$\frac{e^2}{m} \left\langle p\hat{D} \right\rangle_0 = \frac{\hbar\tau}{\pi} \int d\omega\, \omega^3 \int dt' \cos\omega(t-t') I(t-t') \tag{15}$$

with

$$I(t-t') = \int dx \int dp\, p\, e^{-\hat{L}(t-t')} \frac{\partial}{\partial p} Q(t') = \int dx \int dp\, p \frac{\partial}{\partial p'} Q(x', p', t'), \tag{16}$$

where x', p' are the position and momentum variables, respectively, which evolve deterministically (under the action of $\hat{L}$) towards their final values $x = x(t)$, $p = p(t)$. Upon integration by parts, and writing $\int dxdp = \int dx'dp'$ to zero order in e^2, we get

$$I(t-t') = \left\langle \frac{\partial p}{\partial p'} \right\rangle_0 = \frac{1}{2i\eta} \left\langle [\hat{x}', \hat{p}] \right\rangle_0 = \frac{m}{\eta} \sum_k \omega_{k0} \left| x_{0k} \right|^2 \cos\omega_{k0}(t-t'). \tag{17}$$

Now we insert (17) into (15) and integrate over time starting at $-\infty$ (with $y = t - t'$), to take into account that energy balance is established after particle and field have interacted for a sufficiently long time, i.e. in the so-called time-asymptotic limit

$$\begin{aligned} \frac{e^2}{m} \left\langle p\hat{D} \right\rangle_0 &= -\frac{\hbar m\tau}{\pi\eta} \sum_k \omega_{k0} \left| x_{0k} \right|^2 \int_0^\infty d\omega\ \omega^3 \int_0^\infty dy \cos\omega y \cos\omega_{k0} y \\ &= -\frac{\hbar m\tau}{2\eta} \sum_k \omega_{k0} \left| x_{0k} \right|^2 \int_0^\infty d\omega\ \omega^3 [\delta(\omega+\omega_{k0}) + \delta(\omega-\omega_{k0})] \end{aligned} \tag{18}$$

For $\omega_{k0} > 0$ (as is the case for the ground state) the first integral is nil, whence

$$\frac{e^2}{m}\left\langle p\hat{D}\right\rangle_0 = -\frac{\hbar m\tau}{2\eta}\sum_k \omega_{0k}^4 \left|x_{0k}\right|^2 . \tag{19}$$

On comparing with Eq. (14) we obtain

$$\eta = \hbar/2, \tag{20}$$

and Eq. (9) becomes the Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{21}$$

Note that the ZPF has played a crucial role in leading to this result. Firstly, it is the source of the Planck constant in this equation, through the spectral energy density given by Eq. (7). Further, a field with energy spectrum proportional to ω^3 (responsible for the ω_{0k}^4 factor in Eq. (19)) is the single one that guarantees *detailed* balance, by ensuring that Eqs. (14) and (19) have exactly the same structure. This means that energy balance holds not only globally but term by term, or for each frequency. This differs essentially from the result obtained for a *classical* multiply periodic system in equilibrium with a radiation field, in which case balance is attained only if the field has a Rayleigh-Jeans spectrum, proportional to ω^2 [8].

2.3 Detailed balance for an excited harmonic oscillator

Let us now consider an atom in an excited state n, with the background field still being in its ground state (the ZPF). Then instead of (14) we have

$$m\tau\left\langle \dot{x}\,\dddot{x}\right\rangle_n = -m\tau\sum_k \omega_{nk}^4 \left|x_{nk}\right|^2 , \tag{22}$$

and instead of (19) (with $\eta = \hbar/2$) we have

$$\frac{e^2}{m}\left\langle p\hat{D}\right\rangle_n = -m\tau\sum_k \omega_{nk}^4 \left|x_{nk}\right|^2 \mathrm{sign}(\omega_{kn}). \tag{23}$$

Whilst in Eq. (22) all terms have the same sign, Eq. (23) contains now a mixture of positive and negative terms. As a result there is a net average loss of energy

$$\frac{d}{dt}\left\langle H\right\rangle_n \equiv \frac{dH_n}{dt} = -m\tau\sum_k \omega_{nk}^4 \left|x_{nk}\right|^2 (1 - \mathrm{sign}(\omega_{kn})) = -2m\tau\sum_{k<n} \omega_{nk}^4 \left|x_{nk}\right|^2 . \tag{24}$$

Hence, as was to be expected, there cannot be detailed balance between the ZPF and the atom in an excited state; in the sole presence of the ZPF, only the ground state ($n = 0$) is in equilibrium.

Let us now assume that also the background field is in an excited state, and inquire whether in this case there can be equilibrium between particle and field. We write the spectral density of the excited field as

$$\rho(\omega) = \rho_0(\omega)\gamma(\omega), \quad \gamma(\omega) \geq 1$$

where the additional contribution

$$\rho_a(\omega) = \rho - \rho_0 = \rho_0(\gamma - 1) \equiv \rho_0\gamma_a \tag{25}$$

can represent an excitation of the background field or an external field. Now observe that the generalized form of Eq. (23) for the case $\gamma(\omega) > 1$,

$$\frac{e^2}{m}\left\langle p\hat{D}\right\rangle_n = -m\tau\sum_k \omega_{nk}^4 \left|x_{nk}\right|^2 \gamma(|\omega_{nk}|)\mathrm{sign}(\omega_{kn}), \tag{26}$$

contains again a mixture of terms with different signs depending on the sign of ω_{kn}, so that

$$\frac{dH_n}{dt} = -m\tau\sum_k \omega_{nk}^4 \left|x_{nk}\right|^2 [1 - \gamma(|\omega_{nk}|)\mathrm{sign}(\omega_{kn})]. \tag{27}$$

When the values of $|\omega_{nk}|$ differ for different k, the positive and the negative terms in this equation cannot compensate each other in general, and detailed balance can therefore not be satisfied. However, if all values of $|\omega_{nk}|$ in Eq. (27) are equal, it may be possible that detailed balance exists with the particle in an excited state n. This is precisely the case of the harmonic oscillator: all $|\omega_{nk}|$ appearing in (27) are equal and moreover coincide with the oscillator frequency ω_0. With $\left|x_{nn+1}\right|^2 = a(n+1)$, $\left|x_{nn-1}\right|^2 = an$, and $\left|x_{nk}\right|^2 = 0$ for $k \neq n \pm 1$, where $a = \hbar/2m\omega_0$, Eqs. (22) and (26) give

$$m\tau\left\langle \dot{x}\,\dddot{x}\right\rangle_n = -\frac{1}{2}\hbar\tau\omega_0^3(2n+1), \quad \frac{e^2}{2m}\left\langle p\hat{D}\right\rangle_n = -\frac{1}{2}\hbar\tau\omega_0^3\gamma(\omega_0),$$

whence the energy gain or loss is given by

$$\frac{dH_n}{dt} = -\frac{1}{2}\hbar\tau\omega_0^3[(2n+1) - \gamma(\omega_0)].$$

Therefore, detailed balance exists between a harmonic oscillator in its excited state n and an (excited) background field with $\gamma(\omega) = 2n+1$. According to Eq. (25), this field has a spectral energy density

$$\rho_n(\omega) = \rho_0(\omega)(2n+1) = \frac{\hbar\omega^3}{2\pi^2c^3}(2n+1), \tag{28}$$

corresponding to an energy per normal mode $\frac{1}{2}\hbar\omega(2n+1)$, equal to the energy of the mechanical oscillators with which it is in equilibrium. This is simply the condition for balance between field and matter oscillators of the same frequency —and a result that links with the Planck distribution in the case of thermal equilibrium (see [6] and section 3.2 below).

3 Spontaneous and induced transitions

3.1 Radiative lifetimes

Equation (27) determines the average rate of energy loss or gain by the mechanical system in an excited state n due to (upward or downward) transitions to states k with $k > n$ and $k < n$, respectively. In order to analyze this equation in detail it is convenient to write $\gamma(\omega) = 1 + \gamma_a(\omega)$, as follows from Eq. (25), and separate the positive from the negative terms,

$$\begin{aligned}\frac{dH_n}{dt} &= -m\tau \sum_k \omega_{nk}^4 \left|x_{nk}\right|^2 \left[1 - (1 + \gamma_a(|\omega_{nk}|))\mathrm{sign}(\omega_{kn})\right] \\ &= m\tau \sum_k \omega_{nk}^4 \left|x_{nk}\right|^2 \left[(\gamma_a)_{\omega_{kn}>0} - (2 + \gamma_a)_{\omega_{kn}<0}\right]. \end{aligned} \tag{29}$$

The first term within brackets in (29) represents the upward transitions (absorptions) and the second one, the downward transitions (emissions). It is clear that upward transitions can take place only when there is an additional field $\gamma_a(\omega_{kn})$ from which the atom may absorb the necessary energy; in other words, the atom does not ('spontaneously') absorb energy from the ZPF. This is an important point that explains, for example, why optical detectors, including photographic plates, are not activated by the vacuum. Emissions, on the other hand, can be either 'spontaneous' (in presence of just the ZPF) or else stimulated by the additional field $\gamma_a(\omega_{kn})$, according to the second term in (29). Notice in particular that the ground state ($n = 0$, $\omega_{kn} > 0$) is absolutely stable against spontaneous transitions.

Let us now relate the coefficients appearing in the various terms of Eq. (29), with the respective Einstein A and B coefficients. By writing the rate of energy change in terms of the latter

$$\frac{dH_n}{dt} = \sum_{k>n} \hbar \left|\omega_{nk}\right| \left[\rho_a(|\omega_{nk}|)B_{kn}\right] - \sum_{k<n} \hbar \left|\omega_{nk}\right| \left[A_{nk} + \rho_a(|\omega_{nk}|)B_{nk}\right] \tag{30}$$

and comparing term by term with (29), we obtain for the 'spontaneous' emission coefficient

$$A_{nk} = \frac{2\tau m}{\hbar} \left|\omega_{nk}\right|^3 \left|x_{nk}\right|^2 = \frac{4e^2 \left|\omega_{nk}\right|^3}{3\hbar c^3} \left|x_{nk}\right|^2 \tag{31}$$

and for the stimulated transition coefficients

$$B_{nk} = B_{kn} = \frac{m\tau \left|\omega_{nk}\right|^4 \left|x_{nk}\right|^2 \gamma_a(|\omega_{nk}|)}{\hbar \left|\omega_{nk}\right| \rho_a(|\omega_{nk}|)} = \frac{4\pi^2 e^2}{3\hbar^2} \left|x_{nk}\right|^2, \tag{32}$$

in full agreement with the respective QED formulas [9],[10].

3.2 Spontaneous decay and the zero-point field

One can frequently find in the literature that all the spontaneous decay is attributed to either the vacuum fluctuations or radiation reaction, more often to

the latter (see e.g. ([10], [11]-[14]). Let us look at this issue from the perspective of the present theory.

From Eqs. (31) and (32), the ratio of the A to B coefficients is

$$\frac{A_{nk}}{B_{nk}} = \frac{\hbar\,|\omega_{nk}|^3}{\pi^2 c^3} = 2\rho_0(|\omega_{nk}|). \tag{33}$$

Incidentally, this relation and the equality of the coefficients $B_{nk} = B_{kn}$ were predicted by Einstein on the basis of his statistical considerations ([15]; see below).

Notice in particular the factor 2 in equation (33). Given the definition of the coefficients, one could expect the ratio in this equation to correspond exactly to the spectral density of the ZPF, which would mean a factor of 1. However, as follows from Eq. (29), one should actually interpret the factor 2 as $2 = (1 + 1)$. One of these two equal contributions to spontaneous decay is due to the effect of the fluctuations impressed on the particle by the ZPF; the other one is due to Larmor radiation. Their equality (with opposite signs) leads to the exact balance of these two contributions when the particle is in its ground state, thus guaranteeing its stability (cf. Eq. (24) for $n = 0$).

A brief digression is in place regarding the point at which Einstein introduced quantization in his 1917 paper [15], so as to arrive at the Planck distribution. It is frequently argued that he did so through the assumption of discrete atomic levels. However, some time after Einstein's original work, Einstein and Ehrenfest [16] showed that this was not the case, by redoing the calculations with a continuous distribution of atomic levels. In line with the results presented here and in previous work [1], [3] quantization enters through the introduction of a source that includes the ZPF, able to generate 'spontaneous' transitions. This can be easily verified by omitting in the calculation any of the three terms that lead to matter-field equilibrium: stimulated absorptions and emissions, or spontaneous emissions. The absence of the latter leads to absurd results, as happens also with the omission of stimulated absorptions. The omission of the term related to stimulated emissions leads to the expression for the blackbody law proposed by Wien, which correctly approximates Planck's law at low temperatures, so it already contains some quantum principle due to the presence of the term associated with spontaneous emissions. All this can be easily seen in the present context by focusing on just two states n and k, with $\mathcal{E}_n - \mathcal{E}_k = \hbar\omega_{nk} > 0$ and respective populations N_n, N_k. When the system is in thermal equilibrium at temperature T, the relation (k_B is Boltzmann's constant)

$$N_k/N_n = \exp(\mathcal{E}_n - \mathcal{E}_k)/k_B T$$

holds (disregarding inconsequential degeneracies). Since according to Eq. (29) the number of emissions is proportional to $N_n\gamma_a(\omega_{nk})$ and the number of absorptions is proportional to $N_k[2 + \gamma_a(\omega_{nk})]$, from the (detailed) balance condition $N_n\gamma_a = N_k(2 + \gamma_a)$ one obtains indeed Planck's law (for the thermal field) as is well known since 1917 [15],

$$\gamma_a(\omega_{nk}) = \frac{2}{\exp(\mathcal{E}_n - \mathcal{E}_k)/k_B T - 1}. \tag{34}$$

4 Radiative corrections to the energy

The determination of the Lamb shift has been one of the most frequently studied problems in SED and has produced some successful results in the past, though basically restricted to the linear-force problem. Early related works are [17]-[20]; additional references can be seen in [21] and [22]. The theory of SED as developed recently and used in this paper, has the advantage of being applicable to nonlinear forces in general and to the atomic problem in particular. Since this theory includes the radiative terms from the outset, we can use it also to derive general formulas for the radiative energy corrections. Here we present a full derivation of the nonrelativistic atomic Lamb shift and associated effects within the present framework. The results obtained are in line with the predictions deriving from QED; however, the procedure followed for their derivation offers a clear picture of their physical meaning and allows a comparison with alternative interpretations of the Lamb shift found in the literature.

4.1 The Lamb shift

To calculate the energy corrections we go back to the generalized Fokker-Planck equation (11) and multiply it now by xp before integrating over the entire phase space; the result is (assuming again all surface terms to vanish at infinity) [23]

$$\frac{d}{dt}\langle xp\rangle = \frac{1}{m}\left\langle p^2\right\rangle + \langle xf\rangle + m\tau\left\langle x\dddot{x}\right\rangle - e^2\left\langle x\hat{D}\right\rangle. \tag{35}$$

In the radiationless approximation, corresponding to quantum mechanics, the last two terms are neglected and Eq. (35) reduces to

$$\frac{d}{dt}\langle xp\rangle = \frac{1}{m}\left\langle p^2\right\rangle + \langle xf\rangle. \tag{36}$$

Further, in a stationary state (denoted again by n) the time derivative of $\langle xp\rangle$ is zero and Eq. (36) reduces to the virial theorem,

$$\frac{1}{m}\left\langle p^2\right\rangle_n = 2\langle T\rangle_n = -\langle xf\rangle_n, \tag{37}$$

where $\langle T\rangle_n$ is the average kinetic energy. Hence Eq. (35) can be interpreted as a time-dependent version of the virial theorem, with radiative corrections included. Observe that here the average is taken not over time, but over the full particle phase space, which is equivalent to an ensemble average. This is but an example of application of the ergodic properties acquired by the quantum states as discussed in detail in [5], [6].

In the stationary state, the two previously neglected terms can therefore be taken as radiative corrections to the (kinetic) energy,

$$\delta\mathcal{E}_n = \delta\langle T\rangle_n = -\frac{m\tau}{2}\left\langle x\dddot{x}\right\rangle_n + \frac{e^2}{2}\left\langle x\hat{D}\right\rangle_n. \tag{38}$$

This is a general expression for the Lamb shift. Notice that the general laws derived here —such as (12) or (38)— are foreign to quantum theory, where the notion of diffusion (and the related diffusion operator) does not appear at all.

The right´hand side of Eq. (38) will again be calculated to lowest order in e^2, which means calculating the two average values, $\langle x\,\dddot{x}\rangle_n$ and $\left\langle x\hat{D}\right\rangle_n$, to zero order in $\alpha = e^2/\hbar c$. For the first one we get

$$-\frac{m\tau}{2}\langle x\,\ddot{x}\rangle_n = \frac{\tau}{2}\langle \dot{x}\,f\rangle_n = \frac{\tau}{2}\frac{d}{dt}\langle T\rangle_n = 0,$$

which means that the Larmor radiation term does not contribute to the energy shift in the mean, in a stationary state. The correction to the energy comes exclusively from the fluctuations due to the action of the background field on the particle, represented by the second term in Eq. (38). To calculate it we use again Eq. (5), multiply it this time by x,

$$e^2 x\hat{D}(t)Q = \frac{2\hbar}{3\pi c^3}e^2\int d\omega\int dt'\omega^3\cos\omega(t-t')\,x\,e^{-\hat{L}(t-t')}\frac{\partial}{\partial p}Q(t'),$$

and integrate over phase space, thus obtaining

$$\frac{e^2}{2}\left\langle x\hat{D}\right\rangle_n = \frac{\hbar e^2}{3\pi c^3}\int d\omega\,\omega^3\int dt'\cos\omega(t-t')J_n(t-t'), \tag{39}$$

with

$$J_n(t-t') = \int dx\int dp\,x\,e^{-\hat{L}(t-t')}\frac{\partial}{\partial p}Q(t')\Big|_n .$$

Upon an integration by parts (again with $\int dxdp = \int dx'dp'$ to zero order in α) we have $J_n(t-t') = -\langle \partial x/\partial p'\rangle_n$, where

$$J_n(t-t') = \left\langle\frac{\partial x}{\partial p'}\right\rangle_n = \frac{1}{i\hbar}\langle[\hat{x},\hat{x}']\rangle_n = -\frac{2}{\hbar}\sum_k |x_{nk}|^2\sin\omega_{kn}(t-t').$$

Inserting this result into (39) we obtain

$$\frac{e^2}{2}\left\langle x\hat{D}\right\rangle_n = -\frac{2e^2}{3\pi c^3}\sum_k |x_{nk}|^2\int_0^\infty d\omega\;\omega^3\int^t dt'\cos\omega(t-t')\sin\omega_{kn}(t-t'). \tag{40}$$

Extending the initial time integral to $-\infty$ (as corresponds to the time-asymptotic limit) we have (with $y = t-t'$)

$$\int_0^\infty dy\cos\omega y\sin\omega_{kn}y = \tag{41}$$

$$= \frac{1}{2}\int_0^\infty dy\,[\sin(\omega_{kn}+\omega)y + \sin(\omega_{kn}-\omega)y] = \frac{\omega_{kn}}{\omega_{kn}^2-\omega^2},$$

which introduced in Eq. (40) gives for the radiative correction to the (mean kinetic) energy (we write the result in three dimensions, for comparison purposes)

$$\delta\mathcal{E}_n = \frac{e^2}{2}\left\langle \mathbf{x}\cdot\hat{\mathbf{D}}\right\rangle_n = -\frac{2e^2}{3\pi c^3}\sum_k |\mathbf{x}_{nk}|^2\,\omega_{kn}\int_0^\infty d\omega\;\frac{\omega^3}{\omega_{kn}^2-\omega^2}. \tag{42}$$

This result coincides with the formula derived by Power [24] for the Lamb shift on the basis of Feynman's argument [25]. We recall that according to Feynman, the presence of the atom creates a weak perturbation on the nearby field, thereby acting as a refracting medium. The effect of this perturbation is to change the frequencies of the background field from ω to $\omega/n(\omega)$, $n(\omega)$ being the refractive index. The shift of the ZPF energy due to the presence of the atom in state n is then ([10],[24])

$$\Delta\mathcal{E}_n = \sum\frac{1}{2}\frac{\hbar\omega_{kn}}{n_n(\omega_{kn})} - \sum\frac{1}{2}\hbar\omega_{kn} \simeq -\sum[n_n(\omega_{kn})-1]\frac{1}{2}\hbar\omega_{kn},$$

where a summation over the polarizations is included, and the refractive index is given in this approximation by

$$n_n(\omega) \simeq 1 + \frac{4\pi}{3\hbar}\sum_m \frac{|\mathbf{d}_{mn}|^2\,\omega_{mn}}{\omega_{mn}^2-\omega^2}, \tag{43}$$

where $\mathbf{d}_{mn} = e\mathbf{x}_{mn}$ is the transition dipole moment. After an integration over the solid angle and summation over the polarizations, Power obtains in the continuum limit for ω_k the formula

$$\Delta\mathcal{E}_n = -\frac{2}{3\pi c^3}\sum_m |\mathbf{d}_{mn}|^2\,\omega_{mn}\int_0^\infty d\omega\;\frac{\omega^3}{\omega_{mn}^2-\omega^2},$$

which is equal to our result, Eq. (42).

The Lamb shift *proper* (the observable Lamb shift) is obtained by subtracting from the total energy shift given by Eq. (42), the free-particle contribution $\delta\mathcal{E}_{\rm fp}$. This latter is represented by (42) in the limit of continuous electron energies (when ω_{kn} can be ignored compared with ω in the denominator),

$$\delta\mathcal{E}_{\rm fp} = \frac{2e^2}{3\pi c^3}\sum_k |\mathbf{x}_{nk}|^2\,\omega_{kn}\int_0^\infty d\omega\;\omega = \frac{e^2\hbar}{\pi m c^3}\int_0^\infty d\omega\;\omega. \tag{44}$$

To write the last equality we used the sum rule $\Sigma_k\,|\mathbf{x}_{nk}|^2\,\omega_{kn} = 3\hbar/2m$. Since according to Eqs. (6) and (7)

$$\overline{\mathbf{A}^2}^E = \frac{2\hbar}{\pi c}\int_0^\infty d\omega\;\omega, \tag{45}$$

with $\mathbf{A}$ the electromagnetic potential associated with the ZPF, (44) can be rewritten as

$$\delta\mathcal{E}_{\rm fp} = \frac{1}{2m}\overline{\mathbf{A}^2}^E, \tag{46}$$

which identifies this 'free-particle contribution' to the Lamb shift with the contribution from the (free) background field. Its value is independent of the state of the particle; it is a testimony of the ubiquitous presence of the ZPF. Inserting the usual cutoff frequency $\omega_c = mc^2/\hbar$ in the integral in Eq. (44) gives the finite result $\delta\mathcal{E}_{\rm fp} = (\alpha/2\pi)mc^2$.

By subtracting (44) from (42) we obtain for the Lamb shift proper

$$\delta\mathcal{E}_{{\rm L}n} = \delta\mathcal{E}_n - \delta\mathcal{E}_{\rm fp} = -\frac{2e^2}{3\pi c^3}\sum_k |\mathbf{x}_{nk}|^2\,\omega_{kn}^3 \int_0^\infty d\omega\ \frac{\omega}{\omega_{kn}^2-\omega^2}. \tag{47}$$

Inserting once more the cutoff frequency $\omega_c = mc^2/\hbar$ in the integral gives[2]

$$\delta\mathcal{E}_{{\rm L}n} = \frac{2e^2}{3\pi c^3}\sum_k |\mathbf{x}_{nk}|^2\,\omega_{kn}^3 \ln\left|\frac{mc^2}{\hbar\omega_{kn}}\right|, \tag{48}$$

which is Bethe's well-known expression [26].

An important difference between the procedures used in the present paper and in QED to arrive at the Lamb shift formula concerns the mass renormalization. We recall that in the QED case, second-order perturbation theory is used, with the interaction Hamiltonian given by $\hat{H}_{\rm int} = -(e/mc)\hat{\mathbf{A}}\cdot\hat{\mathbf{p}}$. But the energy derived from this term, namely [10]

$$-\frac{2e^2}{3\pi c^3}\sum_k |\mathbf{x}_{nk}|^2\,\omega_{kn}^2 \int_0^\infty d\omega\ \frac{\omega}{\omega-\omega_{nk}},$$

still contains the (linearly divergent) free-particle contribution

$$-\frac{2e^2}{3\pi c^3}\sum_k |\mathbf{x}_{nk}|^2\,\omega_{kn}^2 \int_0^\infty d\omega = -\frac{4e^2}{3\pi c^3}\left(\frac{1}{2m}\sum_k |\mathbf{p}_{nk}|^2\right)\int_0^\infty d\omega$$

that must be subtracted to obtain the Lamb shift proper. Because this result is proportional to the mean kinetic energy, the ensuing correction is taken to represent a mass renormalization,

$$\delta m = \frac{4e^2}{3\pi c^3}\int_0^\infty d\omega, \tag{49}$$

[2]Note that to get a correct (and finite) result, it is essential to leave in the denominator of this formula the term $\tau^2\omega_{mn}^4 \simeq \tau^2\omega^4$ under resonance due to the presence of radiation reaction ([17], [27]). This is a natural term in both QED and SED.

which with the usual cutoff $\omega_c = mc^2/\hbar$ becomes $\delta m = (4\alpha/3\pi)m$.

By contrast, in the derivation presented here to obtain $\delta\mathcal{E}_{\mathrm{L}n}$, Eq. (47), there was no mass renormalization. The result (49) is just the *classical* contribution to the mass predicted by the Abraham-Lorentz equation ([22], Eq. 3.114) (or Maxwell's equations). In the equations of motion (1) this contribution has been already subtracted, so there is no more need to renormalize the mass. However, and as is well known, the formula (47) (common to both SED and renormalized QED) still has a logarithmic divergence that can be remedied by introducing the cutoff frequency ω_c, as was done by Bethe, thus obtaining a very satisfactory result for the Lamb shift.

4.2 Alternative interpretations of the Lamb shift

The interpretation of the Lamb shift as a change of the atomic energy levels due to the interaction with the surrounding ZPF is fully in line with the present theory. It constitutes one additional manifestation of the influence of the particle on the field, which is then fed back on the particle. An alternative way of looking at this reciprocal influence is by considering the general relation between the atomic polarizability α and the refractive index of the medium affected by it, which for $n(\omega) \simeq 1$ can be written as follows,

$$n(\omega) = 1 + 2\pi\alpha(\omega).$$

Comparing this expression with Eq. (43) we obtain

$$\alpha_n(\omega) = \frac{4\pi}{3\hbar}\sum_m \frac{|\mathbf{d}_{mn}|^2\,\omega_{mn}}{\omega_{mn}^2 - \omega^2}, \tag{50}$$

which is the Kramers-Heisenberg formula [11]. This indicates that the Lamb shift can also be viewed as a (second-order in $e\mathbf{E}$) Stark shift associated with the dipole moment $\mathbf{d}(\omega) = \alpha(\omega)\mathbf{E}$ induced by the electric component of the ZPF on the atom.

Equation (47) can be further recast in an (approximate) form that is usual to find in textbooks and Lamb-shift calculations, by assuming that the integral depends so weakly on the index k that such dependence can be ignored. Expressing (47) in terms of energy levels, with $\hbar\omega_{nk} = \mathcal{E}_n - \mathcal{E}_k$, one gets then

$$\begin{aligned}
\delta\mathcal{E}_{\mathrm{L}n} &= -\frac{2e^2}{3\pi c^3}\sum_k |\mathbf{x}_{nk}|^2\,\omega_{kn}^3 \int_0^\infty d\mathcal{E}\;\frac{\mathcal{E}}{(\mathcal{E}_k - \mathcal{E}_n)^2 - \mathcal{E}^2} \\
&= -\frac{2\alpha I_n}{3\pi c^2 m^2}\sum_k |\mathbf{p}_{nk}\cdot\mathbf{p}_{kn}|^2\,(\mathcal{E}_k - \mathcal{E}_n)\,,
\end{aligned}$$

which after a series of transformations becomes [10]

$$\delta\mathcal{E}_{\mathrm{L}n} = \frac{2\alpha I_n}{3\pi c^2 m^2} i\hbar\,\langle n|\,\nabla\hat{V}\cdot\hat{\mathbf{p}}\,|n\rangle = \frac{\alpha\hbar^2 I_n}{3\pi c^2 m^2}\,\langle n|\left[\nabla^2\hat{V}\right]|n\rangle\,.$$

This result suggests to interpret the Lamb shift as due to the variations of the potential energy originating in the fluctuations in $\mathbf{x}$-space. For the Coulomb potential, $\nabla^2 V = 4\pi Z e^2 \delta^3(\mathbf{x})$, so only the wave function at the origin (s states) contributes to the Lamb shift in this approximation. This makes this formula particularly practical for numerical calculations.

4.3 External effects on the radiative corrections

From the results obtained above it is clear that certain basic properties of the vacuum — such as the intensity of its fluctuations or its spectral distribution — are directly reflected in the radiative corrections. This means that a change in such properties can in principle lead to an observable modification of these corrections. The background field can be altered, for instance, by raising the temperature of the system, by adding external radiation, or by introducing objects that affect the distribution of the normal modes of the field.

Such external or 'environmental' effects have been studied for over six decades, normally within the framework of quantum theory, although some calculations have been made also within SED, again for the linear-force (or single-frequency) problem only, leading to comparable results (see [28],[29]). The results of the previous sections, by contrast, can be applied to the general case, without restricting the calculations to the linear-force problem. In the following we present an illustrative selection of results derived for both lifetimes and energy levels. The task is facilitated by the use of the present theory because the influence of the background radiation field (and its modifications) is clearly pictured from the beginning.

In section 3.1 we have already come across one observable effect of a change in the background field: according to Eq. (29) the rates of stimulated atomic transitions are directly proportional to the spectral distribution of the external (or additional) field, be it a thermal field or otherwise. In the case of a thermal field in particular, with $\gamma_a(|\omega_{nk}|)$ given by Eq. (34), the (induced) transition rate from state n to state k becomes (using Eqs. (30) and (32))

$$\frac{dN_{nk}}{dt} = \rho_0(|\omega_{nk}|)\gamma_a(|\omega_{nk}|)B_{nk} = \frac{4e^2\,|\omega_{nk}|^3\,|x_{nk}|^2}{3\hbar c^3}\frac{1}{e^{\hbar|\omega_{nk}|/k_BT}-1}. \qquad (51)$$

This result shows that no eigenstate is stable at $T > 0$ — as is well known — because the thermal field induces both upward and downward transitions. For downward transitions ($\omega_{nk} > 0$) we can rewrite Eq. (51) for comparison purposes in terms of A_{nk} as given by Eq. (31), obtaining

$$\frac{dN_{nk}}{dt} = \frac{A_{nk}}{e^{\hbar|\omega_{nk}|/k_BT}-1}. \qquad (52)$$

At room temperature ($k_BT \simeq .025$ eV) the effect of the thermal field on the decay rate is barely noticeable, since for typical atomic frequencies the inverse of the denominator, $(\exp\hbar\,|\omega_{nk}|\,/k_BT-1)^{-1}$, ranges between $\exp(-40)$ and $\exp(-400)$. The decay of excited states is therefore mostly spontaneous in this

case. For the thermal field to have a noticeable effect on the decay rate, the temperature would have to be of the order of 10^4 K, at which other effects on the atom (assuming it still exists at this high temperature) cannot be ignored.

When the geometry or the spectral distribution of the field is modified by the presence of conducting objects (such as metallic plates or the walls of a cavity), the transition rates are affected accordingly. Let us assume, for simplicity, that the modified field is still isotropic, with the density of modes of a given frequency $|\omega_{nk}|$ reduced by a factor $\gamma_a(|\omega_{nk}|) < 1$: then according to the results of section 3.1 the corresponding spontaneous and induced transition rates are reduced by precisely this factor, since *both* A and ρB are proportional to the density of modes.[3] By enclosing the atoms in a high-quality cavity that excludes the appropriate modes one can therefore virtually inhibit the corresponding transition. For the more general (anisotropic) case the calculations are somewhat more complicated, without however leading to a substantial difference from a physical point of view. Such cavity effects have been the subject of a large number of fine experimental tests since the early works of Kleppner and others within QED ([30]-[33]; for more recent work see, e.g., [34],[35]).

For illustration purposes, let us also briefly indicate how Eqs. (44) and (47) can be used to calculate the changes in the atomic energy shifts produced by the addition of an (external or thermal) field. As in Eq. (25), we denote by $\rho_a = \rho_0 \gamma_a$ the spectral (energy) density of the additional field. The formulas for the variations of the (first-order) radiative corrections are readily obtained by determining the shifts produced by the total field $(\rho_0 + \rho_a)$ and subtracting the original shifts produced by the ZPF alone. The results are

$$\Delta\left(\delta\mathcal{E}_{\mathrm{fp}}\right) = \frac{e^2\hbar}{\pi m c^3} \int_0^\infty d\omega \; \gamma_a \omega, \tag{53}$$

$$\Delta\left(\delta\mathcal{E}_{\mathrm{Ln}}\right) = -\frac{2e^2}{3\pi c^3} \sum_k \left|\mathbf{x}_{nk}\right|^2 \omega_{kn}^3 \int_0^\infty d\omega \; \gamma_a \frac{\omega}{\omega_{kn}^2 - \omega^2}, \tag{54}$$

for a homogeneous, isotropic field. If the additional field represents blackbody radiation at temperature T, γ_a is given by Eq. (34), i.e. $\gamma_a(T) = 2/(\exp y - 1)$ with $y = (\hbar\omega/k_B T)$, and we obtain from Eq. (53)

$$\Delta_T\left(\delta\mathcal{E}_{\mathrm{fp}}\right) = \frac{2\alpha}{\pi m c^2} (k_B T)^2 \int_0^\infty dy \; \frac{y}{\exp y - 1}, \tag{55}$$

whence the free-particle energy increases by the amount

$$\Delta_T\left(\delta\mathcal{E}_{\mathrm{fp}}\right) = \frac{\pi\alpha}{3mc^2} (k_B T)^2. \tag{56}$$

[3]Interestingly, however, by virtue of this proportionality, the ratio of spontaneous to induced transition rates is not altered by a modification of the density of modes.

The formula for the change in the Lamb shift is given from Eq. (54) by

$$\Delta\left(\delta\mathcal{E}_{\mathrm{L}n}\right)=-\frac{4e^{2}}{3\pi c^{3}}\sum_{k}\left|\mathbf{x}_{nk}\right|^{2}\omega_{kn}^{3}\int_{0}^{\infty}d\omega\ \frac{\omega}{\omega_{kn}^{2}-\omega^{2}}\left(\frac{1}{e^{\hbar\omega/kT}-1}\right).$$

These results coincide with those obtained through considerably more cumbersome procedures within QED [36],[37], and the corresponding thermal shifts have also been experimentally confirmed ([38]; see also [39]). From the point of view of SED (or QED) their interpretation is clear: they represent additional contributions to the kinetic energy impressed on the particle by the thermal field, according to the discussion at the beginning of section 4.1.

5 Concluding remarks

All results contained in this paper for a nonrelativistic spinless particle point to the zero-point radiation field as the source not only of quantum behavior itself, but also of the radiative effects on quantum systems. The stochastic problem posed by the action of the ZPF on the particle led to a generalized Fokker-Planck equation for the particle, and all results presented here have been directly derived from this equation. Although as a result of the matter-ZPF interaction both matter and field end up quantized, for the present calculations of the radiative corrections (to lowest order in α) it sufficed to consider the ZPF as a classical function. Indeed, the theory furnishes an alternative way to derive self-consistently results usually considered to be the exclusive province of QED. Important advantages of the present procedure are physical transparency and simplicity; there was no need to resort to heuristic arguments along the derivations. These advantages are particularly apparent in the calculation of environmental effects on the atomic lifetimes and energy levels.

It is important to stress that the present theory implies quantization of both matter and radiation field [1], [2], [6]. This is the ultimate reason that guarantees its equivalence with QED. The difference between these two theories lies not in their final results, but in the whole conceptual picture and the gained clarification of the physics. The present approach gives well-defined answers to deep questions, such as the origin and ultimate meaning of the Schrödinger equation and other puzzles of quantum theory. The consideration of the ZPF as a fundamental ingredient of the theory is thus not a subterfuge conceived to simplify or guide the calculations, but a fundamental step to unfold the deep meaning of the quantum behaviour of matter and field.

However, it should also be stressed that there can be differences. The present theory gives, by construction, only an approximate description of nature. It could be that its further development in search of a more detailed or refined description leads to discrepancies, open to resolution only by experiment.

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